HAMILTONIAN S^1-MANIFOLDS OF DIMENSION $2n$ WITH $n+2$ ISOLATED FIXED POINTS

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Abstract. Let (M,ω) be a compact $2n$-dimensional symplectic manifold equipped with a Hamiltonian S^1 action with $n+2$ isolated fixed points. We will see that n must be even. Such an example is $\widetilde{G}_2(\mathbb{R}^{n+2})$, the Grassmanian of oriented 2-planes in $\mathbb{R}^{n+2}$ with n even, equipped with a standard S^1 action. We show that if the S^1 representations at the fixed points on M are the same as those of the standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$, then the integral cohomology ring and total Chern class of M are the same as those of $\widetilde{G}_2(\mathbb{R}^{n+2})$; on the other hand, if M has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$, then the S^1 representations at the fixed points are the same as those of the standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$.

In particular, if M is Kähler and the action is holomorphic, then any of the following 3 conditions implies that M is equivariantly biholomorphic and symplectomorphic to $\widetilde{G}_2(\mathbb{R}^{n+2})$: (1) M has the same first Chern class as $\widetilde{G}_2(\mathbb{R}^{n+2})$, (2) M has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$, and (3) the S^1 representations at the fixed points are the same as those of the standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$.

1. Introduction

Let S^1 act on a compact $2n$-dimensional symplectic manifold (M,ω) with moment map $\phi\colon M\to\mathbb{R}$. Since $\dim H^{2i}(M;\mathbb{R})\geq 1$ for all $0\leq 2i\leq 2n$, and ϕ is a perfect Morse-Bott function, the above action has at least $n+1$ fixed points. In [7], we studied the case when the action has exactly $n+1$ isolated fixed points. Such examples are some standard S^1 actions on $\mathbb{CP}^n$ and on $\widetilde{G}_2(\mathbb{R}^{n+2})$, the Grassmanian of oriented 2-planes in $\mathbb{R}^{n+2}$ with $n\geq 3$ odd. Let M' denote $\mathbb{CP}^n$ or $\widetilde{G}_2(\mathbb{R}^{n+2})$ with n odd in the latter. Then the following conditions are equivalent: M has the same first Chern class as M', M has the same integral cohomology ring as M', M has the same total Chern class as M', and the S^1 representations at the fixed points are the same as those of a standard S^1 action on M'. If M is Kähler and the action is holomorphic, then any one of these conditions implies that M is equivariantly biholomorphic and symplectomorphic to M' equipped with a standard S^1 action.

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In this paper, we consider a Hamiltonian S^1 action on a compact $2n$-dimensional symplectic manifold with $n+2$ isolated fixed points. The motivation comes from the work in [7] and the following example.

Example 1.1. Let $\widetilde{G}_2(\mathbb{R}^{n+2})$ be the Grassmanian of oriented 2-planes in $\mathbb{R}^{n+2}$, with $n \geq 2$ even. This $2n$ dimensional manifold naturally arises as a coadjoint orbit of $SO(n+2)$, hence it has a natural Kähler structure and a Hamiltonian $SO(n+2)$ action.

Consider the S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$ induced by the S^1 action on $\mathbb{R}^{n+2} = \mathbb{C}^{\frac{n+2}{2}}$ given by

$$\lambda \cdot \left(z_0, z_1, \cdots, z_{\frac{n}{2}}\right) = \left(\lambda^{b_0} z_0, \lambda^{b_1} z_1, \cdots, \lambda^{b_{\frac{n}{2}}} z_{\frac{n}{2}}\right),$$

where the b_i's, with $i = 0, 1, \cdots, \frac{n}{2}$, are mutually distinct integers. This action has $n+2$ isolated fixed points P_0, P_1, $\cdots$, and P_{n+1}, where for each i, P_i and P_{n+1-i} are given by the plane $(0, \cdots, 0, z_i, 0, \cdots, 0)$ respectively with two different orientations. Let ϕ be the moment map of this S^1 action. Then the moment map values of the P_i's are respectively $-b_0, \cdots, -b_{\frac{n}{2}}, b_{\frac{n}{2}}, \cdots, b_0$, assuming in the order of nondecreasing. The set of weights of the action at P_i is

$$\{w_{ij}\} = \{\phi(P_j) - \phi(P_i)\}_{j \neq i, n+1-i}.$$

The manifold $\widetilde{G}_2(\mathbb{R}^{n+2})$ is also a complex quadratic hypersurface in $\mathbb{CP}^{n+1}$.

Our first two main results are as follows.

Theorem 1.2. *Let S^1 act on a compact $2n$-dimensional symplectic manifold (M, ω) in a Hamiltonian fashion with $n+2$ isolated fixed points. Then n must be even. If the S^1 representations at the fixed points are the same as those of a standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$, then the integral cohomology ring and total Chern class of M are the same as those of $\widetilde{G}_2(\mathbb{R}^{n+2})$.*

Theorem 1.3. *Let S^1 act on a compact $2n$-dimensional symplectic manifold (M, ω) in a Hamiltonian fashion with $n+2$ isolated fixed points. If M has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$, then the S^1 representations at the fixed points are the same as those of a standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$.*

For a Hamiltonian S^1 action on a compact $2n$-dimensional symplectic manifold (M, ω) with $n+2$ isolated fixed points, the moment map values of the fixed points are "almost mutually distinct" (see Lemma 2.1). If they are mutually distinct, and if $[\omega]$ is an integral class, then M admits a quasi-ample complex line bundle (see [7, Sect. 4]). By Hattori's work, under an additional assumption, we have the implication: if M has the same first Chern class as $\widetilde{G}_2(\mathbb{R}^{n+2})$, then the S^1 representations at the fixed points are the same as those of a standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$ (see [3, Theorem 6.17 and Proposition 6.26]). Hence with Hattori's additional assumption, Hattori's work and our Theorems 1.2 and 1.3 give us a similar "4 equivalent conditions" as for the case when there are $n+1$ isolated fixed points.

In the case when M is Kähler and the action is holomorphic, we have the following result.

Theorem 1.4. *Let the circle act holomorphically and in a Hamiltonian fashion on a compact Kähler manifold M of complex dimension n with $n+2$ isolated fixed points. Then any one of the following conditions implies that M is equivariantly biholomorphic and symplectomorphic to $\widetilde{G}_2(\mathbb{R}^{n+2})$, equipped with a standard S^1 action.*

1. *M has the same first Chern class as $\widetilde{G}_2(\mathbb{R}^{n+2})$,*
2. *M has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$, and*
3. *the S^1 representations at the fixed points are the same as those of a standard S^1 action on $\widetilde{G}_2(\mathbb{R}^{n+2})$.*

The organization of the paper is as follows. In Section 2, we prove a preliminary lemma for the next sections. In Section 3, we prove Theorem 1.2, and in Section 4, we prove Theorem 1.3. Each of these two sections contains a preliminary part for the proof of the main theorem. In Section 1.4, we prove Theorem 1.4.

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2. Compact Hamiltonian S^1-manifolds of dimension $2n$ with $n+2$ isolated fixed points

Let (M,ω) be a compact symplectic manifold equipped with a symplectic circle action. There exist S^1-invariant almost complex structures $J\colon TM \to TM$ which are **compatible** with ω, i.e., $\omega(J(\cdot),\cdot)$ is an invariant Riemannian metric. The set of such structures on (M,ω) is contractible. Assume the fixed points of the S^1 action are isolated. Then at each fixed point P, there is a well defined set of nonzero integers, called the **weights** of the action; and the normal bundle to P naturally splits into subbundles, one corresponding to each weight. If the S^1 action on M is Hamiltonian with moment map $\phi\colon M \to \mathbb{R}$, then ϕ is a perfect Morse function, with critical points being the fixed points of the action. At each fixed point P, the negative normal bundle to P is the subbundle with negative weights, and the positive normal bundle to P is the subbundle with positive weights. If λ_P is the number of negative weights (counted with multiplicities) at P, then the **Morse index at** P (for the Morse function ϕ) is $2\lambda_P$, it is the dimension of the negative normal bundle to P. Similarly, the **Morse coindex at** P is $2n-2\lambda_P$.

Lemma 2.1. *Let the circle act on a compact $2n$-dimensional symplectic manifold (M,ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume the fixed point set consists of $n+2$ isolated points, $P_0, \cdots, P_{\frac{n}{2}}, P_{\frac{n}{2}+1}, \cdots, P_{n+1}$. Then n must be even, and for each i with $0 \leq i \leq \frac{n}{2}$, P_i has Morse index $2i$, and for each*

i with $\frac{n}{2}+1 \leq i \leq n+1$, P_i has Morse index $2i-2$, and

$$\phi(P_0) < \cdots < \phi(P_{\frac{n}{2}}) \leq \phi(P_{\frac{n}{2}+1}) < \phi(P_{\frac{n}{2}+2}) < \cdots < \phi(P_{n+1}). \tag{2.2}$$

Moreover, the cohomology groups of M are

$$H^k(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even, } 0 \leq k \leq 2n, \text{ and } k \neq n, \\ \mathbb{Z}\oplus\mathbb{Z}, & \text{if } k = n, \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

For a CW-structure of M, the negative disk bundle of a fixed point of index $2i$ is a $2i$-cell, it contributes to $H^{2i}(M;\mathbb{Z})$.

Proof. Since M is compact and symplectic, $\dim H^{2i}(M) \geq 1$ for all $0 \leq 2i \leq 2n$. The moment map is a perfect Morse function, whose critical points — the fixed points of the S^1 action, all have even indices. Hence there is at least one fixed point of index $2i$ for each $0 \leq 2i \leq 2n$. By assumption, there is a remaining fixed point, let $2k$ be its Morse index. Since $\dim H^{2i}(M) = \dim H^{2n-2i}(M)$ for all $0 \leq 2i \leq 2n$ by Poincare duality, the remaining fixed point forces that $2k = 2n - 2k$, which implies that $2k = n$, i.e., n is even, and the remaining fixed point has Morse index n. We name the fixed points as P_0, P_1, $\cdots$, P_{n+1}, where for each i with $0 \leq i \leq \frac{n}{2}$, P_i has Morse index $2i$, and for each i with $\frac{n}{2}+1 \leq i \leq n+1$, P_i has Morse index $2i-2$.

By the following Lemma 2.3, we have $\phi(P_0) < \phi(P_1) < \cdots < \phi(P_{\frac{n}{2}}) \leq \phi(P_{\frac{n}{2}+1})$. Using Lemma 2.3 for the reversed circle action and $-\phi$, we have $-\phi(P_{n+1}) < -\phi(P_n) < \cdots < -\phi(P_{\frac{n}{2}+1}) \leq -\phi(P_{\frac{n}{2}})$. The two inequalities give (2.2).

By Morse theory, M has a natural CW-structure — its cells are the negative disk bundles of the fixed points, which are all even dimensional. By cellular cohomology theory, we have the claimed cohomology groups of M and the contributions to these groups. □

Lemma 2.3. [10, Lemma 3.1] *Let the circle act on a compact symplectic manifold (M,ω) with moment map $\phi\colon M \to \mathbb{R}$. Given any fixed component F, we have*

$$2\lambda_F \leq \sum_{\phi(F')<\phi(F)} \big(\dim(F') + 2\big),$$

where $2\lambda_F$ is the Morse index at F, and the sum is over all fixed components F' such that $\phi(F') < \phi(F)$.

3. Proof of Theorem 1.2

In this section, we will use equivariant cohomology to prove Theorem 1.2. Let us first set up the technical tools needed.

3.1. **Equivariant cohomology and equivariant Chern classes.**

For a smooth S^1-manifold M, the **equivariant cohomology** of M in a coefficient ring R is $H^*_{S^1}(M;R) = H^*(S^\infty \times_{S^1} M; R)$, where S^1 acts on S^∞ freely. If P is a point, then $H^*_{S^1}(P;R) = H^*(\mathbb{CP}^\infty; R) = R[t]$, where $t \in H^2(\mathbb{CP}^\infty; R)$ is a generator. If S^1 acts on M trivially, i.e., it fixes M, then $H^*_{S^1}(M;R) = H^*(M;R) \otimes R[t] = H^*(M;R)[t]$. The projection map $\pi\colon S^\infty \times_{S^1} M \to \mathbb{CP}^\infty$ induces a pull back map

$$\pi^*\colon H^*(\mathbb{CP}^\infty; R) \to H^*_{S^1}(M;R),$$

so that $H^*_{S^1}(M;R)$ becomes an $H^*(\mathbb{CP}^\infty;R)$ module.

Let (M,ω) be a compact Hamiltonian S^1-manifold with moment map $\phi\colon M \to \mathbb{R}$. Assume the fixed point set M^{S^1} consists of isolated points. So M^{S^1} has no torsion cohomology. Let P be a fixed point. Let

$$M_\pm = \{x \in M \mid \phi(x) < \phi(P) \pm \epsilon\},$$

where ϵ is small, and assume P is the only fixed point in $M_+ - M_-$. By Morse theory, M_+ has the homotopy type of M_- glued with the negative disk bundle of P. Consider the long exact sequence in equivariant cohomology with $\mathbb{Z}$ coefficients for the pair (M_+, M_-):

$$\cdots \to H^*_{S^1}(M_+, M_-; \mathbb{Z}) \to H^*_{S^1}(M_+;\mathbb{Z}) \to H^*_{S^1}(M_-;\mathbb{Z}) \to \cdots .$$

Let Λ_P^- be the product of the weights on the negative normal bundle N_P^- to P. The **equivariant Euler class $\mathbf{e^{S^1}(N_P^-)}$ of $\mathbf{N_P^-}$**, is equal to $\Lambda_P^- t^{\lambda_P}$. In the above sequence, $H^*_{S^1}(M_+, M_-;\mathbb{Z}) \cong H^{*-2\lambda_P}_{S^1}(P;\mathbb{Z})$, and the map to $H^*_{S^1}(M_+;\mathbb{Z})$ is multiplication by $\widetilde{e}^{S^1}(N_P^-)$, where $\widetilde{e}^{S^1}(N_P^-)$ is the equivariant extension to M of $e^{S^1}(N_P^-)$. The fact that M^{S^1} has no torsion cohomology makes the map $H^*_{S^1}(M_+, M_-;\mathbb{Z}) \to H^*_{S^1}(M_+;\mathbb{Z})$ injective (for all degrees $*$'s), hence the long exact sequence splits into a short exact sequence

$$0 \to H^*_{S^1}(M_+, M_-;\mathbb{Z}) \to H^*_{S^1}(M_+;\mathbb{Z}) \to H^*_{S^1}(M_-;\mathbb{Z}) \to 0. \tag{3.1}$$

The sequence (3.1) is a crucial starting point to prove the following "injectivity" and "surjectivity" theorems. The inclusion map $\iota\colon M^{S^1} \to M$ induces an injection

$$\iota^*\colon H^*_{S^1}(M;\mathbb{Z}) \hookrightarrow H^*_{S^1}(M^{S^1};\mathbb{Z}), \tag{3.2}$$

and the natural restriction map from equivariant cohomology to ordinary cohomology is a surjection

$$H^*_{S^1}(M;\mathbb{Z}) \twoheadrightarrow H^*(M;\mathbb{Z}). \tag{3.3}$$

The "injectivity" (3.2) means that an equivariant cohomology class is determined by its restriction to the fixed point set. The "surjectivity" (3.3) and the Leray-Hirsch theorem imply that the kernel of the map (3.3) is the ideal generated by $\pi^*(t)$. Hence, to compute the ordinary cohomology of M, it is enough to determine the equivariant cohomology of M as an $H^*(\mathbb{CP}^\infty, \mathbb{Z})$ module. The "injectivity" and "surjectivity" theorems for cohomology in $\mathbb{Q}$

coefficients were given by Kirwan [4], and in $\mathbb{Z}$ coefficients, by Tolman and Weitsman [11]. The preliminary sections in [10] and [8] gave more details on the arguments.

By similar arguments, we can have a basis of $H^*_{S^1}(M;\mathbb{Z})$ as an $H^*(\mathbb{CP}^\infty;\mathbb{Z})$ module as follows.

Proposition 3.4. *Let the circle act on a compact symplectic manifold (M,ω) with moment map $\phi\colon M\to\mathbb{R}$. Assume the fixed points are isolated. Then for each fixed point P of index $2\lambda_P$, there exists a class $\alpha_P\in H^{2\lambda_P}_{S^1}(M;\mathbb{Z})$ such that*

$$\alpha_P|_P=\Lambda^-_P t^{\lambda_P}=e^{S^1}(N^-_P),\ \text{and}$$

$$\alpha_P|_{P'}=0 \text{ for any other fixed point } P' \text{ with } \phi(P')<\phi(P).$$

Moreover, such classes α_P's with $P\in M^{S^1}$ form a basis for $H^*_{S^1}(M;\mathbb{Z})$ as an $H^*(\mathbb{CP}^\infty;\mathbb{Z})$ module.

The following corollary is slightly different from [10, Corollary 2.3].

Corollary 3.5. *Let the circle act on a compact symplectic manifold (M,ω) with moment map $\phi\colon M\to\mathbb{R}$. Assume the fixed points are isolated. Let $\beta\in H^*_{S^1}(M;\mathbb{Z})$ be a class such that $\beta|_P=0$ for all the fixed points P with $\phi(P)<a$ for some $a\in\mathbb{R}$. Then*

$$\beta=\sum_{\phi(P)\geq a,\deg(\alpha_P)\leq\deg(\beta)} a_P\alpha_P,$$

where α_P is the basis element as in Proposition 3.4, the sum is over all the fixed points P's with $\phi(P)\geq a$ and $\deg(\alpha_P)\leq\deg(\beta)$, and $a_P\in H^*_{S^1}(P;\mathbb{Z})\cong H^*(\mathbb{CP}^\infty;\mathbb{Z})$.

Proof. By Proposition 3.4, we can write

$$\beta=\sum_{P\in M^{S^1}} a_P\alpha_P.$$

Restricting this equality to the fixed points in the order that their moment map values are nondecreasing, we get inductively that $a_P=0$ for all P with $\phi(P)<a$. Then consider the degrees of both sides, we obtain the claim. $\square$

Now let (M,ω) be a compact $2n$-dimensional symplectic S^1-manifold with isolated fixed points. Let P be a fixed point, and $\{w_1,w_2,\cdots,w_n\}$ be the set of weights at P. We denote the **equivariant total Chern class of** M as

$$c^{S^1}(M)=1+c^{S^1}_1(M)+\cdots+c^{S^1}_n(M)\in H^*_{S^1}(M;\mathbb{Z}), \tag{3.6}$$

with $c^{S^1}_i(M)\in H^{2i}_{S^1}(M;\mathbb{Z})$ the i-th equivariant Chern class of M. The restriction of $c^{S^1}_i(M)$ to P is

$$c^{S^1}_i(M)|_P=\sigma_i(w_1,\cdots,w_n)t^i, \tag{3.7}$$

where $\sigma_i(w_1, \cdots, w_n)$ is the i-th symmetric polynomial in the weights at P. In particular, if $\mathbf{e^{S^1}(N_P)}$ denotes the **equivariant Euler class of** the normal bundle $\mathbf{N_P}$ to P, then

$$e^{S^1}(N_P) = c_n^{S^1}(M)|_P = \big(\prod_i w_i\big)t^n.$$

We will denote the total Chern class of M as

$$c(M) = 1 + c_1(M) + \cdots + c_n(M) \in H^*(M;\mathbb{Z}),$$

with $c_i(M) \in H^{2i}(M;\mathbb{Z})$ the i-th Chern class of M.

Finally, the projection $\pi\colon S^\infty \times_{S^1} M \to \mathbb{CP}^\infty$ induces a natural push forward map $\pi_*\colon H^*_{S^1}(M;\mathbb{Q}) \to H^*(\mathbb{CP}^\infty;\mathbb{Q})$, which is given by "integration over the fiber M", denoted $\int_M$. We will use the following theorem due to Atiyah-Bott, and Berline-Vergne [1, 2].

Theorem 3.8. *Let the circle act on a compact manifold M. Assume the fixed points are isolated. Fix a class $\alpha \in H^*_{S^1}(M;\mathbb{Q})$. Then as elements of $\mathbb{Q}(t)$,*

$$\int_M \alpha = \sum_{P\subset M^{S^1}} \frac{\alpha|_P}{e^{S^1}(N_P)},$$

where the sum is over all the fixed points.

3.2. **Proof of Theorem 1.2.**

Let (M,ω) be a compact $2n$-dimensional Hamiltonian S^1-manifold with moment map $\phi\colon M \to \mathbb{R}$. Assume the fixed point set consists of $n+2$ isolated points, P_0, P_1, $\cdots$, P_{n+1}, with Morse indices as in Lemma 2.1. We set up the following notations.

- $\mathbf{\Gamma_i}$: the sum of the weights of the S^1 action on the normal bundle to P_i;
- $\mathbf{\Lambda_i}$: the product of the weights of the S^1 action on the normal bundle to P_i;
- $\mathbf{\Lambda_i^-}$: the product of the negative weights of the S^1 action on the normal bundle to P_i;
- $\mathbf{\Lambda_i^+}$: the product of the positive weights of the S^1 action on the normal bundle to P_i.

We first prove the following results.

Proposition 3.9. *Assume we have the above assumptions. Then as a $H^*(\mathbb{CP}^\infty;\mathbb{Z})$ module, $H^*_{S^1}(M;\mathbb{Z})$ has a basis $\{1, \alpha_1, \cdots, \alpha_{n+1}\}$, such that the restrictions of these classes to the fixed points are determined by the weights of the S^1-action as follows.*

For each i with $0 \le i \le \frac{n}{2}$, we have

$$\alpha_i|_{P_i} = \Lambda_i^- t^i,\ \ \alpha_i|_{P_k} = 0 \textit{ for all } k \textit{ with } k < i, \textit{ and} \tag{3.10}$$

$$\alpha_i|_{P_k} = \Lambda_i^- \prod_{j<i} \frac{\Gamma_k - \Gamma_j}{\Gamma_i - \Gamma_j} t^i \textit{ for each } k \textit{ with } k > i. \tag{3.11}$$

For each i *with* $\frac{n}{2}+1 \leq i \leq n+1$, *we have*

$$\alpha_i|_{P_i} = \Lambda_i^- t^{i-1},\ \alpha_i|_{P_k} = 0 \textit{ for all } k \textit{ with } k < i, \textit{ and} \tag{3.12}$$

$$\alpha_i|_{P_k} = -\frac{\Lambda_k}{\Lambda_i^+} \prod_{j>i, j\neq k} \frac{\Gamma_i - \Gamma_j}{\Gamma_k - \Gamma_j} t^{i-1} \textit{ for eack } k \textit{ with } k > i. \tag{3.13}$$

Proof. We use Proposition 3.4 to construct the basis.

First, consider the case $1 \leq i \leq \frac{n}{2}$. Note that the class

$$\alpha_i = \Lambda_i^- \prod_{j<i} \frac{c_1^{S^1}(M) - \Gamma_j t}{\Gamma_i - \Gamma_j}$$

satisfies (3.10), by Proposition 3.4, we can take these α_i's as basis elements. (Thanks [10, Proposition 3.9 or Corollary 3.14] for the forms of these classes.) Restricting α_i to P_k with $k > i$, we get (3.11).

Next we consider the case $\frac{n}{2}+1 \leq i \leq n+1$. For $i = \frac{n}{2}+1$, by Proposition 3.4 and (2.2), there exists a basis element $\alpha_{\frac{n}{2}+1} \in H^n_{S^1}(M;\mathbb{Z})$ such that

$$\alpha_{\frac{n}{2}+1}|_{P_{\frac{n}{2}+1}} = \Lambda^-_{\frac{n}{2}+1} t^{\frac{n}{2}}, \text{ and } \alpha_{\frac{n}{2}+1}|_{P_k} = 0 \text{ for all } k \text{ with } 0 \leq k < \tfrac{n}{2}.$$

If $\phi\big(P_{\frac{n}{2}}\big) < \phi(P_{\frac{n}{2}+1})$, then $\alpha_{\frac{n}{2}+1}|_{P_{\frac{n}{2}}} = 0$ also holds by Proposition 3.4. If $\phi\big(P_{\frac{n}{2}}\big) = \phi(P_{\frac{n}{2}+1})$, let $M_{\frac{n}{2}-1} = \big\{x \in M \mid \phi(x) < \phi\big(P_{\frac{n}{2}}\big)\big\}$. Since $P_{\frac{n}{2}}$ and $P_{\frac{n}{2}+1}$ are of the same Morse index, and are disjoint, we may first glue the negative disk bundle of $P_{\frac{n}{2}}$ to $M_{\frac{n}{2}-1}$, and let M_- be the resulting space. Then we glue the negative disk bundle of $P_{\frac{n}{2}+1}$ to M_-, and let M_+ be the resulting space. Then (3.1) implies that $\alpha_{\frac{n}{2}+1}|_{P_{\frac{n}{2}}} = 0$. Hence (3.12) holds for $i = \frac{n}{2}+1$. For any other i with $\frac{n}{2}+1 < i \leq n+1$, by Proposition 3.4 and (2.2), the basis elements satisfying (3.12) exist. It remains to find $\alpha_i|_{P_k}$ for $k > i$, or prove (3.13). Note that the class

$$\Big(\alpha_i \cdot \prod_{j>i, j\neq k} \big(c_1^{S^1}(M) - \Gamma_j t\big)\Big)$$

has degree less than $2n = \dim(M)$. Using Theorem 3.8 to integrate this class on M, we get

$$0 = \frac{\alpha_i|_{P_i} \cdot \prod_{j>i,j\neq k}(\Gamma_i - \Gamma_j)}{\Lambda_i} + \frac{\alpha_i|_{P_k} \cdot \prod_{j>i,j\neq k}(\Gamma_k - \Gamma_j)}{\Lambda_k}.$$

Solving this and using (3.12), we get (3.13). □

Proof of Theorem 1.2. Let $M' = \widetilde{G}_2(\mathbb{R}^{n+2})$. By assumption, there is a bijection $f\colon M^{S^1} \to (M')^{S^1}$ such that as bundles equipped with S^1 actions, at each fixed point P, $f_*(T_P M) \cong T_{f(P)} M'$. By Proposition 3.9, f induces an

isomorphism $H^*_{S^1}(M;\mathbb{Z})|_P = f^*\big(H^*_{S^1}(M';\mathbb{Z})|_{f(P)}\big)$ for all $P \in M^{S^1}$, and by (3.6) and (3.7), f also induces an isomorphism $c^{S^1}(M)|_P = f^*\big(c^{S^1}(M')|_{f(P)}\big)$ for all $P \in M^{S^1}$. By injectivity (3.2), we have $H^*_{S^1}(M;\mathbb{Z}) = f^*\big(H^*_{S^1}(M';\mathbb{Z})\big)$ as $H^*(\mathbb{CP}^\infty;\mathbb{Z})$ modules, and $c^{S^1}(M) = f^*\big(c^{S^1}(M')\big)$. By surjectivity (3.3), we have isomorphisms $H^*(M;\mathbb{Z}) = f^*\big(H^*(M';\mathbb{Z})\big)$ as rings, and $c(M) = f^*\big(c(M')\big)$. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. First, we give some preliminary results needed.

4.1. **Preliminaries for the Proof of Theorem 1.3.**

First, we have the following equivariant extension $\widetilde{u}$ of the symplectic class $[\omega]$ for a Hamiltonian S^1-space.

Lemma 4.1. [8, Lemma 2.7] *Let the circle act on a connected compact symplectic manifold (M,ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume $[\omega]$ is an integral class. Let F_0 be a fixed component (for example, the minimum of ϕ). Then there exists $\widetilde{u} \in H^2_{S^1}(M;\mathbb{Z})$ such that for any fixed component F,*

$$\widetilde{u}|_F = [\omega|_F] + t\,(\phi(F_0) - \phi(F))\,.$$

For a Hamiltonian S^1-manifold M, when $H^2(M;\mathbb{R}) = \mathbb{R}$, we can express $c_1(M)$ as follows.

Lemma 4.2. [6, Lemma 2.3] *Let the circle act on a connected compact symplectic manifold (M,ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume $H^2(M;\mathbb{R}) = \mathbb{R}$. Then*

$$c_1(M) = \frac{\Gamma_F - \Gamma_{F'}}{\phi(F') - \phi(F)}[\omega],$$

where F and F' are any two fixed components such that $\phi(F') \neq \phi(F)$, and Γ_F and $\Gamma_{F'}$ are respectively the sums of the weights at F and F'.

For a symplectic S^1-manifold, when there exists a finite stabilizer group $\mathbb{Z}_k \subset S^1$, where $k > 1$, the set of points, $M^{\mathbb{Z}_k} \subsetneq M$, which is fixed by $\mathbb{Z}_k$ but not fixed by S^1, is a symplectic submanifold, called an **isotropy submanifold**. If an isotropy submanifold is a sphere, it is called an **isotropy sphere**.

Lemma 4.3. [7, Lemma 2.2] *Let the circle act on a connected compact symplectic manifold (M,ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume $[\omega]$ is integral. Then for any two fixed components F and F', $\phi(F) - \phi(F') \in \mathbb{Z}$. If $\mathbb{Z}_k$ fixes any point on M, then for any two fixed components F and F' on the same connected component of the isotropy submanifold $M^{\mathbb{Z}_k}$, we have $k \,|\, (\phi(F') - \phi(F))$.*

In this section, we assume that M has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$ with n even. This ring structure is as follows.

Lemma 4.4. [9, Theorem 5.1] *The integral cohomology ring of $\widetilde{G}_2(\mathbb{R}^{n+2})$ with n even is as follows. The generators are: (we are using 3 different forms for the last $\frac{n}{2}$ generators.)*

$$1, x, \cdots, x^{\frac{n}{2}-1}, y, z, \text{ and}$$

$$xy = xz = \frac{1}{2}x^{\frac{n}{2}+1}, x^2y = x^2z = \frac{1}{2}x^{\frac{n}{2}+2}, \cdots, x^{\frac{n}{2}}y = x^{\frac{n}{2}}z = \frac{1}{2}x^n,$$

where $\deg(x) = 2$, *and* $\deg(y) = \deg(z) = n$. *More relations are:* $x^{\frac{n}{2}} = y + z$, *and when* $n = 4k+2$, $\frac{1}{2}x^n = yz$, *and* $y^2 = z^2 = 0$*; when* $n = 4k$, $\frac{1}{2}x^n = y^2 = z^2$, *and* $yz = 0$.

Before going to the proofs, let us define the following spaces.

Definition 4.5. Let the circle act on a compact symplectic manifold (M, ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume the fixed point set consists of $n+2$ isolated points, P_0, P_1, $\cdots$, and P_{n+1}. By Lemma 2.1, we may take real numbers a_i with $i = -1, 0, \cdots, \frac{n}{2}-1, \frac{n}{2}+1, \cdots, n+1$ such that

$$a_{-1} < \phi(P_0) < a_0 < \phi(P_1) < a_1 < \cdots < \phi(P_{\frac{n}{2}-1}) < a_{\frac{n}{2}-1} < \phi(P_{\frac{n}{2}}) \leq \phi(P_{\frac{n}{2}+1})$$

$$< a_{\frac{n}{2}+1} < \cdots < \phi(P_{n+1}) < a_{n+1}.$$

Define

$$\mathbf{M_i} = \{x \in M \mid \phi(x) < a_i\}.$$

Let $M_{\frac{n}{2}}$ be the space obtained by gluing the negative disk bundle $D^-_{\frac{n}{2}}$ of $P_{\frac{n}{2}}$ to $M_{\frac{n}{2}-1}$, denoted

$$\mathbf{M_{\frac{n}{2}}} = M_{\frac{n}{2}-1} \cup D^-_{\frac{n}{2}}.$$

Since the negative disk bundle $D^-_{\frac{n}{2}+1}$ of $P_{\frac{n}{2}+1}$ have the same dimension as $D^-_{\frac{n}{2}}$, the gluing of the boundary of $D^-_{\frac{n}{2}+1}$ to $M_{\frac{n}{2}}$ cannot be surjective to $D^-_{\frac{n}{2}}$, hence the gluing of $D^-_{\frac{n}{2}+1}$ is homotopic to a gluing of $D^-_{\frac{n}{2}+1}$ to $M_{\frac{n}{2}-1}$. Let $M_{\frac{n}{2}+1}$ be the resulting space, denoted

$$\mathbf{M_{\frac{n}{2}+1}} = M_{\frac{n}{2}-1} \cup D^-_{\frac{n}{2}+1}.$$

Define similarly

$$\mathbf{M'_i} = \{x \in M \mid \phi(x) > a_{i-1}\}, \mathbf{M'_{\frac{n}{2}+1}} = M'_{\frac{n}{2}+2} \cup D^+_{\frac{n}{2}+1}, \text{ and}$$

$$\mathbf{M'_{\frac{n}{2}}} = M'_{\frac{n}{2}+2} \cup D^+_{\frac{n}{2}},$$

where $D^+_{\frac{n}{2}+1}$ and $D^+_{\frac{n}{2}}$ are respectively the positive disk bundles to $P_{\frac{n}{2}+1}$ and $P_{\frac{n}{2}}$.

4.2. **Proof of Theorem 1.3.**

In this subsection, assume we have the following assumption.

Assumption 4.6. Let the circle act on a compact symplectic manifold (M, ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume the fixed point set consists of $n+2$ isolated points, P_0, P_1, $\cdots$, P_{n+1}. Assume $[\omega]$ is primitive integral, and M has the same integral cohomology ring as $\widetilde{G}_2(\mathbb{R}^{n+2})$ (n is even by Theorem 1.2).

First of all, since $[\omega]$ is integral, by Lemma 4.3, for any i and j, $\phi(P_i) - \phi(P_j) \in \mathbb{Z}$. We have the following results, which follow from Lemmas 4.22, 4.25, and 4.31.

Proposition 4.7. *Assume Assumption 4.6 holds. Then the set of weights at each fixed point P_i is*

$$\{w_{ij}\} = \{\phi(P_j) - \phi(P_i)\}_{j\neq i, n+1-i}.$$

Moreover, for each i with $0 \leq i \leq \frac{n}{2} - 1$,

$$\phi(P_i) - \phi(P_{\frac{n}{2}}) = -\big(\phi(P_{n+1-i}) - \phi(P_{\frac{n}{2}+1})\big).$$

For any i and j, $\phi(P_i) - \phi(P_j)$ depends on the cohomology class of ω. When the symplectic classes on M and on $\widetilde{G}_2(\mathbb{R}^{n+2})$ are the same multiple of the generators, the conclusions of Proposition 4.7 imply Theorem 1.3.

The idea of proof of Proposition 4.7 is as follows. We first obtain the product of the negative weights at each fixed point, then we obtain the set of negative weights at each fixed point. We do the same for the positive weights at each fixed point.

Lemma 4.8. *Under Assumption 4.6, for each i with $0 \leq i \leq \frac{n}{2}$, the product of the negative weights at P_i is*

$$\Lambda_i^- = \prod_{j<i} \big(\phi(P_j) - \phi(P_i)\big), \tag{4.9}$$

and the product of the negative weights at $P_{\frac{n}{2}+1}$ is

$$\Lambda^-_{\frac{n}{2}+1} = \prod_{j<\frac{n}{2}} \big(\phi(P_j) - \phi(P_{\frac{n}{2}+1})\big). \tag{4.10}$$

Proof. By assumption and Lemma 4.4, for i with $0 \leq i \leq \frac{n}{2}$, M_i has the same integral cohomology ring as $\mathbb{CP}^i$. Since $[\omega]$ is primitive integral, we may take $1, [\omega], \cdots, [\omega]^i$ as the generators of the integral cohomology ring of M_i (strictly speaking, their restrictions to M_i). Let $\widetilde{u}$ be as in Lemma 4.1. Since

$$\prod_{j<i} \big(\widetilde{u} + (\phi(P_j) - \phi(P_0))t\big)|_{P_j} = 0 \text{ for all } j < i,$$

By Corollary 3.5,

$$\prod_{j<i}\big(\widetilde{u}+(\phi(P_j)-\phi(P_0))t\big)=c\,\alpha_i \text{ if } i<\tfrac{n}{2}, \text{ and for } i=\tfrac{n}{2},$$

$$\prod_{j<\frac{n}{2}}\big(\widetilde{u}+(\phi(P_j)-\phi(P_0))t\big)=c\,\alpha_{\frac{n}{2}}+d\,\alpha_{\frac{n}{2}+1},$$

where α_i and $\alpha_{\frac{n}{2}+1}$ are the basis elements as in Proposition 3.9, c and d are constants since the degrees of the classes on both sides of the equations are the same. Restricting these equations to M_i, and using the same notations for $\widetilde{u}$ and α_i, we get

$$\prod_{j<i}\big(\widetilde{u}+(\phi(P_j)-\phi(P_0))t\big)=c\,\alpha_i \text{ for } 0\le i\le \tfrac{n}{2}. \tag{4.11}$$

By Proposition 3.9 and surjectivity (3.3), in M_i, we may assume the restriction of α_i to ordinary cohomology is $[\omega]^i$. Restricting (4.11) to ordinary cohomology, we get

$$[\omega]^i=c\,[\omega]^i.$$

So

$$c=1.$$

Restricting (4.11) to P_i, by Lemma 4.1 and (3.10), we get (4.9).

The proof of (4.10) is similar to that of (4.9) for $i=\frac{n}{2}$. By assumption, we may assume that the integral cohomology ring of $M_{\frac{n}{2}+1}$ has generators $1,[\omega],\cdots,[\omega]^{\frac{n}{2}}$. Consider the restriction of the class $\prod_{j<\frac{n}{2}}\big(\widetilde{u}+(\phi(P_j)-\phi(P_0))t\big)$ to $M_{\frac{n}{2}+1}$ and use the same arguments as above. □

Reversing the circle action, using $-\phi$ and the arguments of the proofs of Lemma 4.8, we get the following "symmetric" claims to those of Lemma 4.8.

Lemma 4.12. *Under Assumption 4.6, for each i with $\frac{n}{2}+1\le i\le n+1$, the product of the positive weights at P_i is*

$$\Lambda_i^+=\prod_{j>i}\big(\phi(P_j)-\phi(P_i)\big), \tag{4.13}$$

and the product of the positive weights at $P_{\frac{n}{2}}$ is

$$\Lambda_{\frac{n}{2}}^+=\prod_{j>\frac{n}{2}+1}\big(\phi(P_j)-\phi(P_{\frac{n}{2}})\big).$$

Next, we try to get the product of the negative weights at P_i for $\frac{n}{2}+2\le i\le n+1$. First, for $\dim(M)>4$, we write some $\alpha_i|_{P_{i+1}}$ in (3.13) slightly differently as follows.

Lemma 4.14. *Under Assumption 4.6, assume that* $\dim(M)>4$. *Then for each i with $\frac{n}{2}+1\le i\le n$,*

$$\alpha_i|_{P_{i+1}}=\frac{\Lambda_{i+1}^-}{\phi(P_i)-\phi(P_{i+1})}t^{i-1}. \tag{4.15}$$

Proof. By Lemma 2.1, if $\dim(M) > 4$, then $H^2(M;\mathbb{R}) = \mathbb{R}$. By (3.13), Lemma 4.2, and (4.13), we have

$$\alpha_i|_{P_{i+1}} = -\frac{\Lambda_{i+1}}{\Lambda_i^+}\prod_{j>i+1}\frac{\phi(P_j)-\phi(P_i)}{\phi(P_j)-\phi(P_{i+1})}t^{i-1} = \frac{\Lambda_{i+1}^-}{\phi(P_i)-\phi(P_{i+1})}t^{i-1}.$$

□

Lemma 4.16. *Under Assumption 4.6, assume that* $\dim(M) > 4$*. Then for each* i *with* $\frac{n}{2}+2 \leq i \leq n+1$*, the product of the negative weights at* P_i *is*

$$\Lambda_i^- = \frac{\prod_{j<i}\big(\phi(P_j)-\phi(P_i)\big)}{\big(\phi(P_{\frac{n}{2}})-\phi(P_i)\big)+\big(\phi(P_{\frac{n}{2}+1})-\phi(P_i)\big)}. \tag{4.17}$$

Proof. By assumption, for $\frac{n}{2}+2 \leq i \leq n+1$, we may assume that the generator of $H^{2i-2}(M;\mathbb{Z})$ is $\frac{1}{2}[\omega]^{i-1}$. Hence, for the basis elements α_i's with $\frac{n}{2}+2 \leq i \leq n+1$ in Proposition 3.9, we may assume that the restriction of α_i to ordinary cohomology is $\frac{1}{2}[\omega]^{i-1}$.

First, we prove (4.17) for $P_{\frac{n}{2}+2}$. The class $\prod_{j<\frac{n}{2}+1}\Big(\widetilde{u}+\big(\phi(P_j)-\phi(P_0)\big)t\Big)$ has degree $n+2$ and its restriction to any P_j with $j < \frac{n}{2}+1$ is zero. By Corollary 3.5,

$$\prod_{j<\frac{n}{2}+1}\Big(\widetilde{u}+\big(\phi(P_j)-\phi(P_0)\big)t\Big) = a\,\alpha_{\frac{n}{2}+1} + b\,\alpha_{\frac{n}{2}+2}, \tag{4.18}$$

where $\alpha_{\frac{n}{2}+1}$ is also the basis element in Proposition 3.9, and by comparing the degrees of the classes, we see that $a \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ and $b \in H^0(\mathbb{CP}^\infty;\mathbb{Z})$, i.e., a constant. Restricting (4.18) to ordinary cohomology, we get

$$[\omega]^{\frac{n}{2}+1} = b\,\frac{1}{2}[\omega]^{\frac{n}{2}+1}.$$

Hence

$$b = 2.$$

Restricting (4.18) to $P_{\frac{n}{2}+1}$, using Lemma 4.1, and (4.10), we get

$$a = \big(\phi\big(P_{\frac{n}{2}}\big)-\phi(P_{\frac{n}{2}+1})\big)t.$$

Restricting (4.18) to $P_{\frac{n}{2}+2}$, using Lemma 4.1 and (3.12), we get

$$\prod_{j<\frac{n}{2}+1}\Big(\phi(P_j)-\phi(P_{\frac{n}{2}+2})\Big) = a\,\alpha_{\frac{n}{2}+1}|_{P_{\frac{n}{2}+2}} + b\,\Lambda^-_{\frac{n}{2}+2}.$$

Using this, the values of a and b, and (4.15), we get (4.17) for $P_{\frac{n}{2}+2}$.

Using induction, assume we have the claim (4.17) for some P_i with $i \geq \frac{n}{2}+2$. To prove (4.17) for P_{i+1}, for similar reasons as above, we can write

$$\prod_{j<i}\big(\widetilde{u}+\big(\phi(P_j)-\phi(P_0)\big)t\big) = c\,\alpha_i + d\,\alpha_{i+1}, \tag{4.19}$$

where $c \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ and d is a constant. By restricting (4.19) to ordinary cohomology, we get

$$d = 2.$$

By restricting (4.19) to P_i, using (3.12) and the claim (4.17) on Λ_i^-, we get

$$c = \Big(\phi(P_{\frac{n}{2}}) - \phi(P_i) + \phi(P_{\frac{n}{2}+1}) - \phi(P_i)\Big)\, t.$$

Finally, restricting (4.19) to P_{i+1}, using the values of c and d, and (4.15), we get the claim (4.17) for P_{i+1}. □

Using $-\phi$ we can similarly prove the following claim.

Lemma 4.20. *Under Assumption 4.6, assume that* $\dim(M) > 4$. *Then for each i with $0 \leq i \leq \frac{n}{2} - 1$, the product of the positive weights at P_i is*

$$\Lambda_i^+ = \frac{\prod_{j>i}\big(\phi(P_j) - \phi(P_i)\big)}{\big(\phi(P_{\frac{n}{2}}) - \phi(P_i)\big) + \big(\phi(P_{\frac{n}{2}+1}) - \phi(P_i)\big)}.$$

Next, using the product of the weights at the fixed points, we try to obtain the set of weights at the fixed points.

We choose an S^1-invariant compatible almost complex structure J on M, so we have an S^1-invariant Riemannian metric on M. If X_M is the vector field generated by the circle action, then the gradient vector field of the moment map ϕ is

$$\mathrm{grad}(\phi) = JX_M.$$

The S^1 action and the flow of $\mathrm{grad}(\phi)$ together form a $\mathbb{C}^*$-action. The closure of a nontrivial $\mathbb{C}^*$-orbit contains two fixed points, and it is a sphere, called a **gradient sphere**. A **free gradient sphere** is one whose generic point has trivial stabilizer, and a $\mathbb{Z}_{\mathbf{k}}$ **gradient sphere** is one whose generic point has stabilizer $\mathbb{Z}_k \subset S^1$ for some $k > 1$.

Let P be a fixed point on M with Morse index $2k$. On the negative normal bundle D_P^- to P, assume S^1 acts as follows:

$$\lambda \cdot (z_1, z_2, \cdots, z_k) = (\lambda^{w_1} z_1, \lambda^{w_2} z_2, \cdots, \lambda^{w_k} z_k),$$

where $(w_1, w_2, \cdots, w_k)$ are the negative weights at P. The closure of the $\mathbb{C}^*$-orbit through $(0, \cdots, 0, z_i, 0, \cdots, 0)$, where $i = 1, \cdots, k$, has P and another fixed point Q as poles. We call the corresponding gradient sphere a **weight gradient sphere from** P **to** Q. Similarly, we can define weight gradient spheres from Q to P using the positive normal bundle to Q. If a weight gradient sphere from P to Q is also a weight gradient sphere from Q to P, then we say that there is a **weight gradient sphere between** P **and** Q. In particular, a $\mathbb{Z}_k$ gradient sphere is a $\mathbb{Z}_k$ isotropy sphere, so it is a weight gradient sphere between the two poles of the sphere.

In the following lemmas, when we say "gradient sphere", we are implicitly assuming that a suitable almost complex structure is chosen on the manifold.

We will use the following lemma in our proofs.

Lemma 4.21. *Let the circle act on a compact symplectic manifold* (M, ω) *with moment map* $\phi\colon M \to \mathbb{R}$. *Assume the fixed point set* M^{S^1} *consists of isolated points,* P_0, P_1, $\cdots$, P_m *for some* m, *and there exist real numbers* a_0, a_1, $\cdots$, a_{i+1}, $\cdots$, *such that*

$$\phi(P_0) < a_0 < \phi(P_1) < a_1 < \cdots < \phi(P_i) < a_i < \phi(P_{i+1}) < a_{i+1} < \cdots .$$

Assume that in $\overline{M}_i = \{x \in M \mid \phi(x) < a_i\}$, *the fixed points* P_0, P_1, $\cdots$, P_i *respectively have Morse indices* 0, 2, $\cdots$, $2i$, *and there is a weight gradient sphere between any two fixed points in* $\overline{M}_i$. *Assume* P_{i+1} *has Morse index* $2i+2$, *and* $\overline{M}_{i+1} = \{x \in M \mid \phi(x) < a_{i+1}\}$ *has the rational cohomology ring of* $\mathbb{CP}^{i+1}$. *Then there is a weight gradient sphere from* P_{i+1} *to each* P_j *with* $0 \le j \le i$.

Proof. We used this idea in [7]. The spaces $\overline{M}_i$ and $\overline{M}_{i+1}$ are natural CW complexes — their cells are the negative disk bundles of the fixed points in them, and the gluing maps are induced by the flow of $-\mathrm{grad}(\phi)$. The space $\overline{M}_i$ consists of a unique cell in each even dimension $0, 2, \cdots$, up to $2i$, and $\overline{M}_{i+1}$ consists of a unique cell in each even dimension $0, 2, \cdots$, up to $2i+2$. Now, let us only think about the CW-structures of $\overline{M}_i$ and $\overline{M}_{i+1}$. The assumption on $\overline{M}_i$ means that, at each P_j with $0 \le j \le i$, the CW-complex $\overline{M}_i$ is $2i$ dimensional, and any weight gradient sphere coming up from P_{i+1} to P_j will indeed increase the dimension at P_j. (If at a fixed point P_j in $\overline{M}_i$, there is a "non-weight direction", then a gradient sphere coming up from P_{i+1} to P_j may not increase the dimension at P_j.) Since P_{i+1} has Morse index $2i+2$, there are $i+1$ weight gradient spheres from P_{i+1} to the fixed points below P_{i+1}. If two weight gradient spheres have the same south pole P_j for some $0 \le j \le i$, then the CW-complex $\overline{M}_{i+1}$ is at least $2i+4$ dimensional at P_j, contradicting that $\overline{M}_{i+1}$ has the rational cohomology ring of $\mathbb{CP}^{i+1}$. □

First, we obtain the set of negative weights at P_i with $0 \le i \le \frac{n}{2}+1$ as follows.

Lemma 4.22. *Assume Assumption 4.6 holds. Then in each* M_i *with* $0 \le i \le \frac{n}{2}+1$, *there is a weight gradient sphere between any two fixed points. Moreover, for each* i *with* $0 \le i \le \frac{n}{2}$, *the set of negative weights at* P_i *is*

$$\big\{w_{ij}^-\big\} = \big\{\phi(P_j) - \phi(P_i)\big\}_{j<i}, \tag{4.23}$$

and the set of negative weights at $P_{\frac{n}{2}+1}$ *is*

$$\big\{w_{\frac{n}{2}+1,j}^-\big\} = \big\{\phi(P_j) - \phi(P_{\frac{n}{2}+1})\big\}_{j<\frac{n}{2}}. \tag{4.24}$$

Proof. We first show the following claim that we will use. **Claim**: there is a weight gradient sphere from $P_{\frac{n}{2}+2}$ to $P_{\frac{n}{2}}$, and for each i with $\frac{n}{2}+2 \le i \le n+1$, there is a weight gradient sphere from P_i to P_{i-1}. Consider the CW-structure of M given by Morse theory. Since the generator of $H^{n+2}(M; \mathbb{Q})$ has to do with the generators of $H^n(M; \mathbb{Q})$ (the latter is a factor of the

former), the attaching of the $n+2$-cell, the negative disk bundle of $P_{\frac{n}{2}+2}$ to the n-skeleton induced by the flow of $-\text{grad}(\phi)$, cannot miss the n-cells, in particular, cannot miss $P_{\frac{n}{2}}$. Hence there exists a weight gradient sphere from $P_{\frac{n}{2}+2}$ to $P_{\frac{n}{2}}$, and there exists a weight gradient sphere from $P_{\frac{n}{2}+2}$ to $P_{\frac{n}{2}+1}$. For each i with $\frac{n}{2}+3 \leq i \leq n+1$, for a similar reason, that is, the generator of $H^{2i-2}(M;\mathbb{Q})$ has to do with the generator of $H^{2i-4}(M;\mathbb{Q})$ (one is $\frac{1}{2}[\omega]^{i-1}$, and the other is $\frac{1}{2}[\omega]^{i-2}$), there is a weight gradient sphere from P_i to P_{i-1}.

The space M_0 only contains the fixed point P_0, so the claims hold for M_0. Consider M_1, which contains the fixed points P_0 and P_1. The fixed point P_1 has Morse index 2, its negative disk bundle has to flow to P_0, so there is a weight gradient sphere from P_1 to P_0, and (4.9) for P_1 is (4.23) for P_1. By the last paragraph, there is a weight gradient sphere from P_{n+1} to P_n, using $-\phi$, we have equivalently that there is a weight gradient sphere from P_0 to P_1. Since P_1 has index 2, the weight gradient sphere from P_1 to P_0 and the one from P_0 to P_1 must coincide. Now, consider M_2 which contains the fixed points P_0, P_1 and P_2. By assumption, M_2 has the rational cohomology ring of $\mathbb{CP}^2$. By Lemma 4.21, there is a weight gradient sphere from P_2 to each P_j with $j=0,1$. Combining with Lemma 4.3, for $j=0,1$, there is a negative weight w_{2j}^- which divides $\phi(P_j)-\phi(P_2)$, then (4.9) for P_2 gives us the claim (4.23) for P_2. We see that the weight gradient sphere from P_2 to P_0 is an isotropy sphere, hence it is a weight gradient sphere between P_2 and P_0. Similar to the above, by the last paragraph, there is a weight gradient sphere from P_n to P_{n-1}, using $-\phi$, we have equivalently that there is a weight gradient sphere from P_1 to P_2. Since the index of P_2 is 4 and there is a weight gradient sphere from P_2 to P_j for each $j=0,1$, the weight gradient sphere from P_2 to P_1 and the one from P_1 to P_2 must coincide. (There cannot be a non-weight gradient sphere from P_2 to P_0 or P_1.) So in M_2, there is a weight gradient sphere between any two fixed points. Inductively using the above arguments, we obtain the claims for all the M_i's and for all the fixed points involved. $\square$

Reversing the circle action, using $-\phi$, Lemma 4.12 and similar arguments, we can show the following "symmetric" claims to those of Lemma 4.22.

Lemma 4.25. *Assume Assumption 4.6 holds. Then in each M_i' with $\frac{n}{2} \leq i \leq n+1$, there is a weight gradient sphere between any two fixed points. Moreover, for each i with $\frac{n}{2}+1 \leq i \leq n+1$, the set of positive weights at P_i is*

$$\left\{w_{ij}^+\right\} = \left\{\phi(P_j)-\phi(P_i)\right\}_{j>i}, \tag{4.26}$$

and the set of positive weights at $P_{\frac{n}{2}}$ is

$$\left\{w_{\frac{n}{2},j}^+\right\} = \left\{\phi(P_j)-\phi(P_{\frac{n}{2}})\right\}_{j>\frac{n}{2}+1}. \tag{4.27}$$

In particular, for $\dim(M)=4$, Lemmas 4.22 and 4.25 give all the weights.

Corollary 4.28. *Assume Assumption 4.6 holds and* $\dim(M) = 4$. *Then the set of weights at the fixed points are as follows.*

$$\begin{aligned}
&At\ P_0: \ w_{01} = \phi(P_1) - \phi(P_0), \ and\ w_{02} = \phi(P_2) - \phi(P_0).\\
&At\ P_1: \ w_{10} = \phi(P_0) - \phi(P_1), \ and\ w_{13} = \phi(P_3) - \phi(P_1).\\
&At\ P_2: \ w_{20} = \phi(P_0) - \phi(P_2), \ and\ w_{23} = \phi(P_3) - \phi(P_2).\\
&At\ P_3: \ w_{31} = \phi(P_1) - \phi(P_3), \ and\ w_{32} = \phi(P_2) - \phi(P_3).
\end{aligned}$$

Moreover,

$$\phi(P_3) - \phi(P_2) = \phi(P_1) - \phi(P_0). \tag{4.29}$$

We can obtain (4.29) by using Theorem 3.8 to integrate 1 on M.

Next, we try to obtain the set of negative weights at the fixed points P_i's with $\frac{n}{2}+2 \leq i \leq n+1$, and the set of positive weights at the fixed points P_i's with $0 \leq i \leq \frac{n}{2}-1$.

Remark 4.30. We had the spaces M_i's in Definition 4.5. In the proof of the following lemma, when we say the "CW-complex M_i", we are only thinking about its CW-structure — its cells and the attaching maps induced by the flow of $-\mathrm{grad}(\phi)$.

Lemma 4.31. *Assume Assumption 4.6 holds. Then for each i with $\frac{n}{2}+2 \leq i \leq n+1$, the set of negative weights at P_i is*

$$\left\{w_{ij}^-\right\} = \left\{\phi(P_j) - \phi(P_i)\right\}_{j<i, j\neq n+1-i}, \tag{4.32}$$

and for each i with $0 \leq i \leq \frac{n}{2}-1$, the set of positive weights at P_i is

$$\left\{w_{ij}^+\right\} = \left\{\phi(P_j) - \phi(P_i)\right\}_{j>i, j\neq n+1-i}. \tag{4.33}$$

Moreover, for each i with $0 \leq i \leq \frac{n}{2}-1$, we have

$$\phi(P_i) - \phi(P_{\frac{n}{2}}) = \phi(P_{\frac{n}{2}+1}) - \phi(P_{n+1-i}). \tag{4.34}$$

Proof. By Corollary 4.28, the claims hold when $\dim(M) = 4$. So we only need to consider $\dim(M) > 4$.

By Lemmas 4.22 and 4.25, there is a weight gradient sphere between $P_{\frac{n}{2}}$ and P_j for any $j \neq \frac{n}{2}+1$, and there is a weight gradient sphere between $P_{\frac{n}{2}+1}$ and P_j for any $j \neq \frac{n}{2}$. Since $\dim(M) = 2n$, there is no weight gradient sphere between $P_{\frac{n}{2}}$ and $P_{\frac{n}{2}+1}$.

We first prove the claims for $P_{\frac{n}{2}+2}$ and $P_{\frac{n}{2}-1}$. Let $D^-_{\frac{n}{2}+2}$ be the negative disk bundle of $P_{\frac{n}{2}+2}$. Then we have the following claim (1a), and we will show the rest.

(1a) there is a weight gradient sphere between $P_{\frac{n}{2}+2}$ and P_j with $j = \frac{n}{2}+1, \frac{n}{2}$.
(1b) the flow down of $D^-_{\frac{n}{2}+2}$ surjects to the CW-complex $M_{\frac{n}{2}-1}$.
(1c) there is no weight gradient sphere from $P_{\frac{n}{2}+2}$ to $P_{\frac{n}{2}-1}$.
(1d) there is a weight gradient sphere from $P_{\frac{n}{2}+2}$ to each P_j with $0 \leq j \leq \frac{n}{2}-2$.

Proof of (1b). Since the generator of $H^{n+2}(M;\mathbb{Q})$ has each of the two generators of $H^n(M;\mathbb{Q})$ as a factor, the flow down of $D^-_{\frac{n}{2}+2}$ must surject to the interiors of the two n-cells, the negative disk bundles of $P_{\frac{n}{2}+1}$ and $P_{\frac{n}{2}}$. By continuity of the flow, the flow down of $D^-_{\frac{n}{2}+2}$ surjects to the intersection of the flow downs of the two n-cells. Consider the CW-structures of $M_{\frac{n}{2}+1}$, $M_{\frac{n}{2}}$, and $M_{\frac{n}{2}-1}$ given by Morse theory, by Lemma 4.22, the intersection of the flow downs of the two n-cells is exactly the CW-complex $M_{\frac{n}{2}-1}$. □

Proof of (1c). By (1a) and (1b), we see that the flow down of $D^-_{\frac{n}{2}+2}$ contains P_j with $j=\frac{n}{2}+2, \frac{n}{2}+1, \frac{n}{2}, \frac{n}{2}-1$. Similarly, the flow up of the positive disk bundle $D^+_{\frac{n}{2}-1}$ of $P_{\frac{n}{2}-1}$ also contains these 4 fixed points. Let M^2_{-1} be the intersection of the flow down of $D^-_{\frac{n}{2}+2}$ and the flow up of $D^+_{\frac{n}{2}-1}$. It contains the above 4 fixed points. Since in M^2_{-1}, there are only two weight gradient spheres from $P_{\frac{n}{2}}$ (and from $P_{\frac{n}{2}+1}$), M^2_{-1} is 4-dimensional. Since there are already two weight gradient spheres from $P_{\frac{n}{2}+2}$, there cannot be a weight gradient sphere from $P_{\frac{n}{2}+2}$ to $P_{\frac{n}{2}-1}$, and the point $P_{\frac{n}{2}-1}$ is the south pole of a non-weight gradient sphere from $P_{\frac{n}{2}+2}$ inside the space M^2_{-1}. □

Proof of (1d). The Morse index of $P_{\frac{n}{2}+2}$ is $n+2$. By (1a), (1b) and (1c), there is a $2(\frac{n}{2}-1)$-dimensional subspace $(D^-_{\frac{n}{2}+2})^s$ of $D^-_{\frac{n}{2}+2}$ which is attached to the CW-complex $M_{\frac{n}{2}-2}$. By assumption, $H^*(M_{\frac{n}{2}-2};\mathbb{Z}) = \mathbb{Z}[x]/x^{\frac{n}{2}-1}$, where $x=[\omega]$. Since $x^{\frac{n}{2}-1}\neq 0$ in $M_{\frac{n}{2}+2}$, the attaching of $(D^-_{\frac{n}{2}+2})^s$ to the CW-complex $M_{\frac{n}{2}-2}$ makes the resulting space to have the rational cohomology ring of $\mathbb{CP}^{\frac{n}{2}-1}$. By Lemma 4.22, in $M_{\frac{n}{2}-2}$, there is a weight gradient sphere between any two fixed points. Using Lemma 4.21 for the attaching of the cell $(D^-_{\frac{n}{2}+2})^s$ to $M_{\frac{n}{2}-2}$, we get that there is a weight gradient sphere from $P_{\frac{n}{2}+2}$ to each P_j with $0\leq j\leq \frac{n}{2}-2$. □

By (1d) and Lemma 4.3, there is a negative weight $w^-_{\frac{n}{2}+2,j}$ at $P_{\frac{n}{2}+2}$ such that

$$w^-_{\frac{n}{2}+2,j} \,\big|\, \big(\phi(P_j)-\phi(P_{\frac{n}{2}+2})\big) \text{ for each } j \text{ with } 0\leq j\leq \tfrac{n}{2}-2. \tag{4.35}$$

By Lemma 4.25,

$$w^-_{\frac{n}{2}+2,j} = \phi(P_j)-\phi\big(P_{\frac{n}{2}+2}\big) \text{ with } j=\tfrac{n}{2}+1 \text{ and } \tfrac{n}{2} \tag{4.36}$$

are negative weights at $P_{\frac{n}{2}+2}$. Combining (4.36), (4.35), (1c), and (4.17) for $i=\frac{n}{2}+2$, we get that the set of negative weights at $P_{\frac{n}{2}+2}$ is

$$\big\{w^-_{\frac{n}{2}+2,j}\big\} = \big\{\phi(P_j)-\phi(P_{\frac{n}{2}+2})\big\}_{0\leq j\leq \frac{n}{2}+1, j\neq\frac{n}{2}-1},$$

moreover,

$$\phi(P_{\frac{n}{2}-1})-\phi(P_{\frac{n}{2}+2}) = \phi(P_{\frac{n}{2}})-\phi(P_{\frac{n}{2}+2})+\phi(P_{\frac{n}{2}+1})-\phi(P_{\frac{n}{2}+2}),$$

which simplifies to (4.34) for $P_{\frac{n}{2}-1}$. Symmetrically, we have similar statements to (1a), (1b), (1c) and (1d) for $D^+_{\frac{n}{2}-1}$ and $P_{\frac{n}{2}-1}$ by using $-\phi$, combining with Lemma 4.20, we get that the set of positive weights at $P_{\frac{n}{2}-1}$ is

$$\{w^+_{\frac{n}{2}-1,j}\} = \{\phi(P_j) - \phi(P_{\frac{n}{2}-1})\}_{\frac{n}{2}\le j\le n+1, j\neq \frac{n}{2}+2}. \tag{4.37}$$

Next we prove the claims for $P_{\frac{n}{2}+3}$ and $P_{\frac{n}{2}-2}$. Let $D^-_{\frac{n}{2}+3}$ be the negative disk bundle of $P_{\frac{n}{2}+3}$. Let us prove the following claims.

(2a) there is a weight gradient sphere between $P_{\frac{n}{2}+3}$ and P_j for each $j = \frac{n}{2}+2, \frac{n}{2}+1, \frac{n}{2}, \frac{n}{2}-1$.
(2b) the flow down of $D^-_{\frac{n}{2}+3}$ surjects to the CW-complex $M_{\frac{n}{2}-2}$.
(2c) there is no weight gradient sphere from $P_{\frac{n}{2}+3}$ to $P_{\frac{n}{2}-2}$.
(2d) there is a weight gradient sphere from $P_{\frac{n}{2}+3}$ to each P_j with $0 \le j \le \frac{n}{2}-3$.

Proof of (2a). The claim for $j = \frac{n}{2}+2, \frac{n}{2}+1, \frac{n}{2}$ follows from Lemma 4.25. By the last paragraph and (4.37), there is a weight gradient sphere between $P_{\frac{n}{2}+3}$ and $P_{\frac{n}{2}-1}$. □

Proof of (2b). Since the generator of $H^{n+2}(M;\mathbb{Q})$ is a factor of the generator of $H^{n+4}(M;\mathbb{Q})$, the flow down of $D^-_{\frac{n}{2}+3}$ surjects to the interior of the $n+2$-cell, $D^-_{\frac{n}{2}+2}$. Then (2b) follows from continuity of the flow and (1b). □

Proof of (2c). By (2a) and (2b), the flow down of $D^-_{\frac{n}{2}+3}$ contains P_j with $j = \frac{n}{2}+3, \frac{n}{2}+2, \frac{n}{2}+1, \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2$. Similarly, the flow up of the positive disk bundle $D^+_{\frac{n}{2}-2}$ of $P_{\frac{n}{2}-2}$ also contains these 6 fixed points. Let M^3_{-2} be the intersection of the flow down of $D^-_{\frac{n}{2}+3}$ and the flow up of $D^+_{\frac{n}{2}-2}$. Since in M^3_{-2}, there are only 4 weight gradient spheres from $P_{\frac{n}{2}}$ (and from $P_{\frac{n}{2}+1}$), M^3_{-2} is 8-dimensional. Since there are already 4 weight gradient spheres from $P_{\frac{n}{2}+3}$, there cannot be a weight gradient sphere from $P_{\frac{n}{2}+3}$ to $P_{\frac{n}{2}-2}$. The point $P_{\frac{n}{2}-2}$ is the south pole of a non-weight gradient sphere from $P_{\frac{n}{2}+3}$ inside M^3_{-2}. □

Proof of (2d). The Morse index of $P_{\frac{n}{2}+3}$ is $n+4$. By (2a), (2b) and (2c), a $2(\frac{n}{2}-2)$-dimensional subspace $(D^-_{\frac{n}{2}+3})^s$ of $D^-_{\frac{n}{2}+3}$ is attached to the CW-complex $M_{\frac{n}{2}-3}$. By assumption, $H^*(M_{\frac{n}{2}-3};\mathbb{Z}) = \mathbb{Z}[x]/x^{\frac{n}{2}-2}$, where $x = [\omega]$. Since $x^{\frac{n}{2}-2} \neq 0$ in $M_{\frac{n}{2}+3}$, the attaching of $(D^-_{\frac{n}{2}+3})^s$ to the CW-complex $M_{\frac{n}{2}-3}$ makes the resulting space to have the rational cohomology ring of $\mathbb{CP}^{\frac{n}{2}-2}$. By Lemma 4.22, in $M_{\frac{n}{2}-3}$, there is a weight gradient sphere between any two fixed points. Using Lemma 4.21 for the attaching of the cell $(D^-_{\frac{n}{2}+3})^s$

to $M_{\frac{n}{2}-3}$, we get that there is a weight gradient sphere from $P_{\frac{n}{2}+3}$ to each P_j with $0 \leq j \leq \frac{n}{2} - 3$. $\square$

The rest of the arguments for proving (4.32) for $i = \frac{n}{2} + 3$ and (4.34) for $i = \frac{n}{2} - 2$ is similar to the last case by using (2a), (2c), (2d) and (4.17) for $i = \frac{n}{2} + 3$. The proof of (4.33) for $i = \frac{n}{2} - 2$ is by symmetry.

Using the arguments inductively, we can prove the claims for all the fixed points involved. $\square$

5. When the manifold is Kähler — Proof of Theorem 1.4

In this section, we prove Theorem 1.4.

By Theorem 1.2, the following lemma holds. Here we give another proof, so that without Theorem 1.2, we can still prove Theorem 1.4.

Lemma 5.1. *Let S^1 act on a compact $2n$-dimensional symplectic manifold (M, ω) with moment map $\phi\colon M \to \mathbb{R}$. Assume $[\omega]$ is an integral class and the fixed point set consists of $n+2$ isolated points, P_0, P_1, $\cdots$, and P_{n+1}. If the S^1 representations at the fixed points are the same as those of a standard S^1 action on $\widetilde{G}_2\left(\mathbb{R}^{n+2}\right)$ with $n \geq 2$ even, as in Example 1.1, then*

$$c_1(M) = n[\omega].$$

Proof. First assume $\dim(M) > 4$. Then by Lemma 2.1, $H^2(M; \mathbb{R}) = \mathbb{R}$. By assumption, the weights of the S^1 action at the fixed points are as in Example 1.1. Using Lemma 4.2, we get $c_1(M)$ as claimed.

Now assume $\dim(M) = 4$. Since α_1 and α_2 of Proposition 3.4 are basis of $H^2_{S^1}(M; \mathbb{Z})$, we may write

$$c_1^{S^1}(M) = a\alpha_1 + b\alpha_2 + ct, \text{ where } a, b, c \in \mathbb{Z}.$$

Restricting this equation respectively to P_0, P_1 and P_2, using Proposition 3.4 (and the weights as in Example 1.1 or more conveniently in Corollary 4.28), we get

$$c = \Gamma_0, a = 2, \text{ and } b = 0.$$

Using Proposition 3.4, Lemma 4.1 (and the weights as in Example 1.1 or Corollary 4.28), we get that

$$\widetilde{u}|_{P_i} = \alpha_1|_{P_i} \text{ for } i = 0, 1, 2.$$

Since $2\lambda_{P_3} > 2$, $\deg(\widetilde{u}) = \deg(\alpha_1) = 2$, by (3.1), we have

$$\widetilde{u} = \alpha_1.$$

So

$$c_1^{S^1}(M) = 2\widetilde{u} + \Gamma_0 t.$$

Restricting this to ordinary cohomology, we get $c_1(M) = 2[\omega]$. $\square$

The following arguments of proof of Theorem 1.4 was used in [6].

Using a theorem by Kobayashi and Ochiai [5], and by incorporating the circle action, we proved the following result, which is part of Proposition 4.2 in [6].

Proposition 5.2. *Let (M, ω, J) be a compact Kähler manifold of complex dimension n, which admits a holomorphic Hamiltonian circle action. Assume that $[\omega]$ is an integral class. If $c_1(M) = n[\omega]$, then M is S^1-equivariantly biholomorphic to a quadratic hypersurface in $\mathbb{CP}^{n+1} = \mathbb{P}\left(H^0(M; L)\right)$, where L is a holomorphic line bundle over M with first Chern class $[\omega]$ and $H^0(M; L)$ is its space of holomorphic sections.*

Proof of Theorem 1.4. We may assume that $[\omega]$ is a primitive integral class. By Theorem 1.3 and Lemma 5.1 (or by Theorems 1.2 and 1.3), we see that any one of the conditions in Theorem 1.4 gives that $c_1(M) = n[\omega]$. By Proposition 5.2, there is an equivariant biholomorphism

$$f\colon (M, \omega, J) \to (\widetilde{G}_2\left(\mathbb{R}^{n+2}\right), \omega_{st}, J_{st}),$$

where $\widetilde{G}_2\left(\mathbb{R}^{n+2}\right)$ is equipped with the standard S^1 action as in Example 1.1, ω_{st} and J_{st} are standard symplectic and complex structures on $\widetilde{G}_2\left(\mathbb{R}^{n+2}\right)$. We may assume that $[\omega_{st}]$ is primitive integral and $[\omega] = [f^*\omega_{st}]$. We consider the family of forms $\omega_t = (1-t)\omega + tf^*\omega_{st}$ on M, where $t \in [0, 1]$. Each ω_t is nondegenerate: for any point $x \in M$, suppose $X \in T_xM$ is such that $\omega_t(X, Y) = 0$ for all $Y \in T_xM$. In particular, if $Y = JX$, then $\omega_t(X, JX) = 0$. Using the facts $\omega(X, JX) \geq 0$, $f_*(JX) = J_{st}f_*X$, and $\omega_{st}(f_*X, J_{st}f_*X) \geq 0$, we get $X = 0$. So ω_t is a family of symplectic forms in the same cohomology class. By Moser's method, we obtain an equivariant symplectomorphism between M and $\widetilde{G}_2\left(\mathbb{R}^{n+2}\right)$. $\square$

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